Received: *September 11, 1998*, Accepted: *November 13, 1998*

$D = 11$ SUGRA as the low energy effective action of matrix theory: three form scattering

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Abstract: We employ the LSZ reduction formula for matrix theory introduced in our earlier work to compute the t-pole S-matrix for three form–three form scattering. The result agrees completely with tree level $D = 11$ SUGRA. Taken together with previous results on graviton-graviton scattering this shows that matrix theory indeed reproduces the bosonic sector of the $D = 11$ SUGRA action including the Chern-Simons term. Furthermore we provide a detailed account of our framework along with the technology to compute any matrix theory one-loop t-pole scattering amplitude at vanishing p^- exchange.

Keywords: M-Theory, D-branes, M(atrix) Theories, Supergravity Models.

Contents

1. Introduction

It is now commonly believed that eleven-dimensional supergravity [1] is the low-energy effective theory of a more fundamental microscopic theory, known as M-theory [2]. A non-perturbative definition of M-theory has been conjectured to be given in terms of the large N limit of a quantum mechanical supersymmetric $U(N)$ Yang-Mills model called matrix theory [3]. Since the time of this conjecture, many computations in various settings have been performed to test the proposal.[1] Most of these works, however, involve the comparison of classical gravity source-probe actions with the background field effective action of the super Yang-Mills quantum mechanical system evaluated on straight line configurations. Clearly, however, a principle test of the conjecture [3] would be to compute the tree level S-matrix of $D = 11$ SUGRA in matrix theory. We began this project in [6] where we found, using a formalism which enabled the computation of

[1] See [4] for an exhaustive list of references.

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true scattering amplitudes in matrix theory, that indeed the $D = 11$ graviton–graviton tree level t-pole S-matrix agrees precisely with that obtained from matrix theory. We stress that what was computed was the full field theoretical amplitude, i.e. some 66 terms depending on physical polarisations and momenta, in contrast to previous works yielding phase shifts in semi-classical eikonal scattering.

In this paper, we turn our attention away from the pure gravity sector of the theory and consider three form scattering.[2] Again, making use of the leading effective potential for D-particles at one-loop [6, 7, 8], we find complete agreement between the matrix theory and $D = 11$ SUGRA S-matrices (at tree level for the t-pole), an amplitude consisting of 103 terms. Together with our previous results on graviton-graviton scattering [6] this computation confirms that matrix theory describes all bosonic three point interactions of the $D = 11$ supergravity action, including the Chern-Simons term $\int dC \wedge dC \wedge C$. In this sense $D = 11$ SUGRA emerges as the low energy effective action of matrix theory.

In addition we give a detailed account of the formalism which could only be sketched in our previous letter [6], allowing one to compute any t-pole zero p^- exchange S-matrix element in Matrix Theory at one loop. Finally, as in [6], throughout the paper we work in the $N = 2$ sector of the matrix model, so that we are considering the Susskind finite N generalisation [9] of the matrix theory conjecture.

The paper is organised as follows. The main idea of our framework is that S-matrix elements can be constructed from the asymptotic particle states of [10] and involve the expectation of the usual matrix theory effective potential (including background fermions) between in and outgoing polarisation states. We refer to this relation between matrix theory effective actions and S-matrix elements as the matrix theory Lehmann, Symanzik and Zimmermann (LSZ) reduction formula and give a detailed derivation of this result in section 2. Although the one-loop matrix theory effective action could be derived using quantum mechanical Feynman graphs it is obtained more efficiently from the one-loop effective potential of two moving D-particles, computed using the Green-Schwarz boundary state formalism [11, 7, 8]. In section 3 we present a systematic derivation of this result. In section 4 we combine the results of sections 2 and 3 to compute the matrix theory three form–three form scattering amplitude. Furthermore we include a detailed account of the matrix theory kinematics involved, along with the algebra of spinor bilinear operators acting on polarisation states, an essential ingredient for efficient computations of such amplitudes. Section 5 presents the $D = 11$ SUGRA tree level computation of the three form-three form scattering amplitude plus its reduction to the vanishing p^- exchange kinematics described by elastic $N = 2$ matrix theory scattering. Finally we give our conclusions and in an appendix spell out some conventions and Fierz identities needed for our computations.

[2] The three form contribution to the linearized $D = 11$ SUGRA potential has been computed in Matrix Theory by [5].

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2. LSZ for matrix theory

The purpose of this section is to provide the link between the canonical operator based matrix theory scattering amplitude calculations of [10] and the "standard" path integral, background field, effective action approach of, e.g., [12, 13]. This provides the technology required for scattering amplitude calculations in matrix theory.

2.1. Asymptotic particle states in matrix theory

We begin with a short review of the asymptotic state analysis of [10]. We shall primarily study the $U(2)$ matrix theory Hamiltonian whose coordinates take values in the adjoint representation of $U(2)$ i.e.,

$$\begin{aligned} X_m &= X^0_m\, i\mathbf{1} + X^A_m\, i\sigma^A\,, \qquad m = 1,\ldots,9\,, \\ \theta &= \theta^0\, i\mathbf{1} + \theta^A\, i\sigma^A\,, \end{aligned} \tag{2.1}$$

where σ^A are the Pauli matrices. Employing a vector notation for the $SU(2)$ part in which $\vec{X}_m = (X^1_m, X^2_m, X^3_m)$ and similarly for $\vec{\theta}$, the matrix theory Hamiltonian is given by[3]

$$H = \tfrac{1}{2}\, P^0_m P^0_m + \Big(\tfrac{1}{2}\, \vec{P}_m \cdot \vec{P}_m + \tfrac{1}{4}\, (\vec{X}_m \times \vec{X}_n)^2 + \tfrac{i}{2}\, \vec{X}_m \cdot \vec{\theta}\, \gamma_m \times \vec{\theta}\Big)\,. \tag{2.2}$$

It must be augmented by the Gauss law constraint

$$\vec{L} = \vec{X}_m \times \vec{P}_m - \frac{i}{2} \vec{\theta} \times \vec{\theta}\,, \tag{2.3}$$

which annihilates physical states. We wish to study particles widely separated in (say) the ninth transverse direction, it is therefore useful (but not necessary), following [14], to choose a frame where $\vec{X}_9$ lies along the z-axis

$$X^1_9 = 0 = X^2_9\,. \tag{2.4}$$

Denoting the Cartan coordinates $x_m = X^3_m$ and the remainder $Y^I_a = X^I_a$ (with $I = 1, 2$ and $a = 1,\ldots,8$) one then sees that the Hamiltonian (2.2) takes the form

$$H = H_{\rm CoM}(X^0) + H_V(x_m) + H_{\rm HO}(Y^I_a, \theta^I | x_m) + H_4(Y^I_a, x_m, \theta^I, \theta^3)\,. \tag{2.5}$$

Here $H_{\rm CoM} = -\frac{1}{2}\, (\partial_{X^0_m})^2$ is the $U(1)$ center of mass Hamiltonian. In particular, one now observes that $H_V = -\frac{1}{2x_9}\, (\partial_{x_m})^2 x_9$ represents free particle propagation along the "Cartan valley", whereas $H_{\rm HO}$ describes a system of 16 superharmonic oscillators transverse to the Cartan valley with frequency $r = (x_m x_m)^{1/2}$ depending on the distance from the valley origin. Finally H_4 constitutes all remaining terms.

[3]Note that we are using a real, symmetric representation of the $SO(9)$ Dirac matrices in which the charge conjugation matrix $C = \mathbf{1}$.

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In [10] it was shown that in the limit $x_9 \to \infty$, which one interprets as the large separation in the nine direction between a pair of asymptotic particles (importantly, observe that in a general frame, $x_9 = (\vec{X}_9 \cdot \vec{X}_9)^{1/2}$ is gauge invariant, i.e. commutes with the constraint), there exists a split of the Hamiltonian into a free and an interacting part (the latter of which is suppressed in the large x_9 limit). The free Hamiltonian admits eigenstates of the form

$$|p^1_m, \mathcal{H}^1; p^2_m, \mathcal{H}^2\rangle = |0_B, 0_F\rangle_{x_m} \, \tfrac{1}{x_9} \, e^{i(p^1-p^2)\cdot x} e^{i(p^1+p^2)\cdot X^0} |\mathcal{H}^1\rangle_{\theta^0+\theta^3} \, |\mathcal{H}^2\rangle_{\theta^0-\theta^3} \,. \qquad (2.6)$$

Here $|0_B, 0_F\rangle_{x_m}$ denotes the superharmonic groundstate of H_{HO} with vanishing zero point energy. Note that the oscillator states depend explicitly on the Cartan coordinates x_m through their frequency r. In the above, p^I_m are the momenta of the two particles and their possible polarisations $\mathcal{H}^I$ are those of the graviton, three-form tensor and gravitino represented by states

$$|h\rangle = h_{mn}|-\rangle^{mn}, \qquad |C\rangle = C_{mnp}|-\rangle^{mnp}, \qquad |\psi\rangle = \psi^{\alpha}_{m}|-\rangle^{m}_{\alpha} \,, \qquad (2.7)$$

whose explicit form was constructed in [10]. Finally, the subscripts $\theta^0 \pm \theta^3$ in (2.6) indicate from which fermionic variables the polarisation states are built. The state (2.6) is free in the asymptotic limit, i.e.,

$$\lim_{x_9\to\infty} H|p^1_m, \mathcal{H}^1; p^2_m, \mathcal{H}^2\rangle = \tfrac{1}{2} \left[(p^1)^2 + (p^2)^2\right] |p^1_m, \mathcal{H}^1; p^2_m, \mathcal{H}^2\rangle \qquad (2.8)$$

for large particle separations x_9. Moreover, the state (2.6) indeed satisfies the physical state condition $\vec{L}\,|p^1_m, \mathcal{H}^1; p^2_m, \mathcal{H}^2\rangle = 0$.

2.2. The LSZ reduction formula for matrix theory

We now turn to the computation of scattering amplitudes and derive, in particular, the formula relating matrix theory effective action computations with the scattering matrix. For a $1+2 \to 4+3$ process one starts with the S-matrix element

$$\begin{aligned} S_{fi} &= \langle p^4, \mathcal{H}^4; p^3, \mathcal{H}^3|\exp\{-iHT\}|p^1, \mathcal{H}^1; p^2, \mathcal{H}^2\rangle \\ &= \int d^9X'^0\, 4\pi {x'_9}^2 d^9x'\; d^9X^0\; 4\pi {x_9}^2 d^9x\, \tfrac{1}{x'_9}\, e^{-i(p^4-p^3)_m x'_m} e^{-i(p_4+p_3)_m X'^0_m} \\ &\qquad {}_{\theta^0+\theta^3}\langle\mathcal{H}_4|\,{}_{\theta^0-\theta^3}\langle\mathcal{H}_3|\;{}_{x'_m}\langle 0_B, 0_F|\,\exp(-iHT)\,|0_B,0_F\rangle_{x_m}\;|\mathcal{H}_1\rangle_{\theta^0+\theta^3}\;|\mathcal{H}_2\rangle_{\theta^0-\theta^3} \\ &\qquad \tfrac{1}{x_9}\, e^{i(p^1-p^2)_m x_m} e^{i(p^1+p^2)_m X^0_m} \,, \end{aligned} \qquad (2.9)$$

where T is the large time during which the process takes place. The measure factors $4\pi {x_9}^2$ appear in the integral because of the choice of frame (2.4). Moreover, since we are interested in eikonal kinematics, we have chosen asymptotic states describing particles widely separated in the nine direction for both the in and out states. More general configurations may be handled via the insertion of a rotation operator in (2.9).

For complete clarity, we note the following. Strictly speaking, one should compute $S_{fi} = \langle \mathrm{out}|\exp(-iHT)|\mathrm{in}\rangle$ with H being the Hamiltonian (2.2) in a general frame and the asymptotic states $|\mathrm{in}\rangle$ and $\langle\mathrm{out}|$ written in a manifestly gauge invariant way *without* fixing the frame $X_9^I = 0$, as shown in [10]. Now as $\vec{L}|\mathrm{in}\rangle = 0 = \langle\mathrm{out}|\vec{L}$ and $[H, \vec{L}] = 0$ we have

$$S_{fi} = \langle \mathrm{out}|\exp(-iHT)|\mathrm{in}\rangle = \langle \mathrm{out}|\Pi\exp(-iHT)\Pi|\mathrm{in}\rangle\,, \tag{2.10}$$

where $\Pi = (\mathrm{vol}_{SU(2)})^{-1}\int d\vec{\Omega}_{SU(2)}\,\exp(i\vec{\Omega}\cdot\vec{L})$ is the projector onto gauge invariant states. Therefore one is able to choose a frame $X'^I_9 = 0 = X_9^I$ at *both* start and endpoints and in this way arrives at (2.9) in which the variables of the fixed frame (2.4) appear.

The vacuum to vacuum transition amplitude ${}_{x'_m}\langle 0_B, 0_F|\,\exp(-iHT)\,|0_B, 0_F\rangle_{x_m}$ is now the object of interest and may be represented as a path integral with appropriate boundary values for the Cartan coordinates

$$\begin{aligned} e^{i\Gamma(x_m, x'_m, \theta^3) - iH_{\mathrm{CoM}}T} &\equiv {}_{x'_m}\langle 0_B, 0_F|\,\exp(-iHT)\,|0_B, 0_F\rangle_{x_m} = \\ &= \mathcal{N}\int_{x_m(-T/2)=x_m,\,\theta^3(-T/2)=\theta^3}^{x_m(T/2)=x'_m,\,\theta^3(T/2)=\theta^3} D^{16}Y D^{16}\theta^I D^9 x_m\,(4\pi {x_9}^2)\exp(i\int_{-T/2}^{T/2} dt L)\,. \end{aligned} \tag{2.11}$$

Here the Lagrangian L is the usual one ("$p\dot{q} - H$") associated with the Hamiltonian (2.5) in the special frame ($X_9^I = 0$). In particular, as a result of this choice of frame we have a measure factor $4\pi {x_9}^2$ at each point in the path which may be exponentiated via ghosts. $\mathcal{N}$ denotes some normalisation factor.[4] Importantly, note that the boundary conditions for the oscillator variables Y_a^I and θ^I are zero at start and endpoints since we are considering a transition between harmonic oscillator ground states in these variables.

It is also essential to observe that the transition element appearing in (2.9) depends on *operator* valued θ^3, whereas in (2.11) one computes a c-number valued path integral. To elevate this result to an operator, as needed in the rest of the computation, one makes only errors proportional to the square of the momentum transfer q^2 which we anyway neglect in this work since they correspond to contact terms not detectable in our $D0$-brane computation. To see that the θ^3 boundary conditions in the transition amplitude (2.11) are correct, one can change from the sixteen real variables θ^3 to eight complex ones and perform a coherent state analysis similar to that of [16].

We now make the following observation. Firstly, consider the BRST gauge fixed path integral of the ten dimensional super Yang-Mills dimensionally reduced to quan-

[4] In fact, the observant reader will note that in what follows, we assume that the normalisation factor $\mathcal{N}$ behaves as $(x'_9\, x_9)^{-1}$ to cancel the measure factors of the initial and final integrations over the valley coordinates x_m. That this is the case can be argued by $SO(9)$ covariance of the final result. We also do not compute the overall normalisation of the path integral. Such technicalities should, in principle, be rigorously calculable via a careful construction of the path integral similar to that presented in [15].

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$$e^{i\Gamma(x_m,x'_m,\theta^3)} = \mathcal{N}\int_{x_m(-T/2)=x_m,\,\theta^3(-T/2)=\theta^3}^{x_m(T/2)=x'_m,\,\theta^3(T/2)=\theta^3} D\vec{A}\,D^9\vec{X}_m\,D^{16}\vec{\theta}\,D\vec{b}\,D\vec{c}\,\times$$
$$\times\exp(i\int_{-T/2}^{T/2} dt\,L_{\mathrm{BRST}})\,, \qquad (2.12)$$

where $\vec{b}$ and $\vec{c}$ are $SU(2)$ ghosts, $L_{\mathrm{BRST}}(\vec{A},\vec{X},\vec{\theta},\vec{b},\vec{c})$ is the dimensionally reduced super Yang-Mills Lagrangian with ghost terms and the gauge field $\vec{A}$ is the time component of the ten dimensional Yang-Mills field. This is the path integral considered in most matrix theory computations (including the boundary conditions quoted in (2.12)). Then, if one takes the gauge choice $A^3 = 0 = X^I_9$ and additionally integrates out the ghosts and remaining gauge field components A^I (yielding, respectively, the measure factor $4\pi x^2$ and the frame fixed Lagrangian), one obtains *exactly* the path integral (2.11). The ramifications of this simple observation are clear; one may now obtain S-matrix elements from the effective actions produced by the usual matrix theory computations found in the literature [4]. The path integral (2.12) can be computed by expanding about classical trajectories $X^3_m \equiv x^{\mathrm{cl}}_m(t) = b_m + v_m t$ and constant $\theta^3(t) = \theta^3$ which satisfy the quoted boundary conditions for impact parameter $b_m = (x'_m + x_m)/2$ and velocity $v_m = (x'_m - x_m)/T$. A quantum mechanical Feynman diagram expansion in the gauge of one's choice then leads to the required effective action although we found it more efficient to exploit the connection between the matrix theory and String Theory D0-brane dynamics in order to obtain $\Gamma(x'_m, x_m, \theta^3)$ (see section 3).

Our LSZ reduction formula for matrix theory, relating the effective action to S-matrix elements is therefore

$$S_{fi} = \int d^9X'^0\,d^9x'\;d^9X^0\,d^9x\,\times$$
$$\times\exp\Big(-i(p^4-p^3)_m x'_m - i(p_4+p_3)_m X'^0_m + i(p^1-p^2)_m x_m + i(p^1+p^2)_m X^0_m\Big)\times$$
$$\times\,{}_{\theta^0+\theta^3}\langle\mathcal{H}_4|\,{}_{\theta^0-\theta^3}\langle\mathcal{H}_3|e^{i\Gamma(x_m,x'_m,\theta^3)-iH_{CoM}T}|\mathcal{H}_1\rangle_{\theta^0+\theta^3}\,|\mathcal{H}_2\rangle_{\theta^0-\theta^3}\,. \qquad (2.13)$$

Finally, we complete this section by noting that the generalisation of this formula to higher N matrix theory, i.e., $SU(N)$, N particle elastic scattering with vanishing p^- momentum exchange is straightforward. One needs only replace the pairs of incoming and outgoing polarisation states in (2.13) by a set of N such states. The effective action becomes that of an $SU(N)$ matrix theory computation depending on $N-1$ Cartan coordinates and the momentum plane waves become those of N in and outgoing particles.

3. String computation of the $D0$-brane effective potential

In this section, after a very brief review of the Green-Schwarz boundary state formalism [11], we give a detailed account of the computation of the one-loop matrix theory potential (see equation (8) of [6]), first performed in [8, 6]. We consider here the D-particle case, but it is clear from [7, 8] that our result is trivially extendable to generic p-branes.

Dp-brane defects [17] can be described in String Theory by suitable boundary states implementing the usual Neumann-Dirichlet boundary conditions, both in the covariant [18, 19, 20] as well as the Green-Schwarz formalism [11]. In the latter framework, the boundary state describing a single flat D-brane is defined by the BPS condition

$$Q^a_+|B\rangle = 0\ ,\ Q^{\dot a}_+|B\rangle = 0\,, \tag{3.1}$$

where $Q^a_+, Q^{\dot a}_+$ are suitable linear combinations of the $SO(8)$ supercharges $Q^a, \widetilde{Q}^a, Q^{\dot a}, \widetilde{Q}^{\dot a}$. The solution for $|B\rangle$ turns out to be

$$|B\rangle = \exp \sum_{n>0} \left(\frac{1}{n} M_{ij}\alpha^i_{-n}\widetilde{\alpha}^j_{-n} - iM_{a\dot a}S^a_{-n}\widetilde{S}^{\dot a}_{-n}\right)|B_0\rangle \tag{3.2}$$

$|B_0\rangle$ being the zero mode part

$$|B_0\rangle = M_{ij}|i\rangle|\widetilde{j}\rangle - iM_{\dot a b}|\dot a\rangle|\widetilde{b}\rangle \tag{3.3}$$

with $M_{ij}, M_{a\dot a}, M_{\dot a a}$ definite $SO(8)$ matrices [11, 7], depending on the dimensionality of the brane.[5] In this gauge, the $\pm$ light-cone directions satisfy automatically Dirichlet boundary conditions, meaning that they are, effectively, Euclidean branes. One might think that defining the boundary state for moving branes by simply boosting the static one [22] would then be problematical, however, as explained in [7, 8], it is possible to overcome this difficulty by identifying one of the $SO(8)$ transverse directions with the time direction. Thereafter one deduces the corresponding $SO(1,9)$ expressions and performs a double analytic continuation to the final covariant result.

Given these preliminaries, one may then compute arbitrary one-loop n-point functions of vertex operators $V_1, \ldots, V_n$

$$\mathcal{A}_n = \int_0^\infty dt\, \langle B, \vec{x}|e^{-2\pi t\alpha' p^+(P^- - i\partial/\partial x^+)}V_1 \ldots V_n|B, \vec{y}\rangle \tag{3.4}$$

with $P^- = \frac{(p^i)^2}{p^+} +$ osc. the light-cone Hamiltonian (the term $i\partial/\partial x^+$ just implements the p^- subtraction needed to obtain the effective Hamiltonian in this gauge). The configuration space boundary state $|B, \vec{x}\rangle$ is given by

$$|B, \vec{x}\rangle = (2\pi\sqrt{\alpha'})^{4-p}\int \frac{d^{9-p}q}{(2\pi)^{9-p}}\, e^{i\vec{q}\cdot\vec{x}}\,|B\rangle \otimes |\vec{q}\rangle \tag{3.5}$$

[5] In writing $M_{a\dot a}$ we have implicitly chosen to work in the IIA theory, relevant for the analysis of D-particles.

with $\langle q|q'\rangle = \mathrm{vol}_{p+1}\,(2\pi)^{9-p}\delta^{(9-p)}(q-q')$ and vol_{p+1} is the space-time volume spanned by the p-brane. Equation (3.4) describes the interaction of D-branes, considered as semiclassical heavy objects, with n arbitrary states, described by the vertex operators V_i. We are interested in computing the leading one-loop effective action $\Gamma(v,\theta_3,r)$ of two moving D-particles with relative velocity v at their minimum separation r, that is, the usual v^4/r^7 term plus all other terms, bilinear in fermions, related to it by supersymmetry. Correspondingly, we will consider the following correlator, encoding in particular the abovementioned terms

$$\mathcal{V} = \frac{1}{2}\int_0^\infty dt\,\langle B,\vec{x}=0|e^{-2\pi t\alpha' p^+(P^- - i\partial/\partial x^+)}e^{(\eta Q_- + \widetilde{\eta}\widetilde{Q}_-)}e^{V_B}|B,\vec{y}=\vec{b}\rangle \tag{3.6}$$

$Q_-,\widetilde{Q}_-$ being the SO(8) supercharges broken by the presence of D-branes and V_B the vertex operator that boosts the branes to a relative velocity v_i, given explicity by

$$V_B = v_i\oint_{\tau=0} d\sigma\left(X^{[1}\partial_\sigma X^{i]} + \frac{1}{2}S\,\gamma^{1i}S\right), \tag{3.7}$$

where the direction 1 will be Wick rotated to give the time direction. The factor 1/2 in (3.6) has been introduced in order to normalise the v^4 term to one.

A configuration of parallel branes preserves 1/2 of the supercharges; in light-cone gauge this implies that among the 16 linearly realised supercharges $S_0^a,\widetilde{S}_0^{\dot{a}}$, eight of them are left unbroken. Equations (3.4) and (3.6) require then the insertion of at least eight zero modes (that, due to the constraint (3.1), can be always chosen to be S_0^a) in order to get a non-vanishing result. This is precisely the number of zero modes provided by the v^4/r^7 term and all its related fermionic terms, which are therefore determined by massless string excitations alone [7, 8]. In this configuration we can then consider only the massless part of (3.6) which simplifies dramatically. Integrating over the cylinder modulus t, we obtain

$$\mathcal{V} = T\,T_0^2\int\frac{d^9q}{(2\pi)^9}\frac{e^{i\vec{q}\cdot\vec{b}}}{q^2}\langle B_0|e^{(\eta Q_-^0 + \widetilde{\eta}\widetilde{Q}_-^0)}e^{V_B^{(F)}}|B_0\rangle\,, \tag{3.8}$$

where $Q_-^0,\widetilde{Q}_-^0$ are just the zero mode part of the given supercharges, $V_B^{(F)}$ is the fermionic part of the boost operator (3.7), $T_0 = \sqrt{\pi}(4\pi^2\alpha')^{3/2}$ is the tension of 0-branes, T is the overall time in which the interaction takes place and $\vec{q}$ spans the directions $\pm$,2,...,8. The bosonic part of V_B induces conservation of momentum that reads $q^1 = \vec{q}\cdot\vec{v}$; we fix in (3.8) and in the following $q^1 = \vec{q}\cdot\vec{v} = 0$, which simply means we are computing the effective potential at the time[6] $t=0$. It is now convenient to write the S_0 zero mode trace in terms of $R_0^{ij} = (S_0^a\gamma_{ab}^{ij}S_0^b)/4$. Expanding the exponentials we

[6]From now on for simplicity we will omit the overall time of the interaction T and the 0-brane tension T_0.

find, following [8],

$$\mathcal{V}_m = \sum_{\substack{n,m:\\ n/2+m=4}} \frac{2^m}{n!m!}\int \frac{d^9q}{(2\pi)^9}\frac{e^{i\vec{q}\cdot\vec{b}}}{q^2} t^{1i_1\ldots 1i_m j_1 j_2\ldots j_{n-1}j_n} \times$$
$$\times v_{i_1}\ldots v_{i_m}(\sqrt{q^+}\eta + \frac{\tilde{\eta}q_l\gamma^l}{\sqrt{q^+}})^{[a_1}\ldots(\sqrt{q^+}\eta + \frac{\tilde{\eta}q_m\gamma^m}{\sqrt{q^+}})^{a_n]}\gamma^{j_1j_2}_{a_1a_2}\ldots\gamma^{j_{n-1}j_n}_{a_{n-1}a_n}\,, \qquad (3.9)$$

where $t^{i_1\ldots i_8}$ is the usual eight-tensor

$$t^{i_1\ldots i_8} \equiv \mathrm{Tr}_{S_0}\, R_0^{i_1i_2}R_0^{i_3i_4}R_0^{i_5i_6}R_0^{i_7i_8}, \qquad (3.10)$$

whose explicit form can be found, e.g., in the appendix of chapter nine, volume II of [23]. Although the explicit computation for the v^4, v^3 and v^2 cases has been already performed in [8], we will report them here for completeness

v^4**-term**:

$$\mathcal{V}_4 = \frac{2^4}{4!}\int \frac{d^9q}{(2\pi)^9}\frac{e^{i\vec{q}\cdot\vec{b}}}{q^2} t^{1i1j1k1l}v_iv_jv_kv_l = G_9(\vec{b})\,v^4, \qquad (3.11)$$

where $G_d(\vec{b})$ is the propagator for a scalar massless particle in d-dimensions.

v^3**-term**:

$$\mathcal{V}_3 = \frac{2^3}{2!3!}\int \frac{d^9q}{(2\pi)^9}\frac{e^{i\vec{q}\cdot\vec{b}}}{q^2} t^{1i1j1klm}v_iv_jv_k(q^+\eta\gamma^{lm}\eta + 2\,q_n\eta\gamma^{lm}\gamma^n\tilde{\eta} + \frac{q_nq_p}{q^+}\tilde{\eta}\gamma^n\gamma^{lm}\gamma^p\tilde{\eta}) =$$
$$= 2\,v^2 v_m\int \frac{d^9q}{(2\pi)^9}\frac{e^{i\vec{q}\cdot\vec{b}}}{q^2}(q^+\eta\gamma^{1m}\eta + 2\,q_n\eta\gamma^{1mn}\tilde{\eta} + \frac{q^2_{l.c.}}{q^+}\tilde{\eta}\gamma^{1m}\tilde{\eta})\,, \qquad (3.12)$$

where $q^2_{l.c.} = \sum_{i=2}^8 q_iq_i$ and $q^\pm = q^0 \pm q^9$; notice that $q^2 = q^2_{l.c.} - q^+q^-$ implying that $q^- = q^2_{l.c.}/q^+$, modulo contact terms that are vanishing in our configuration of separated 0-branes. It is trivial to verify that the $SO(1,9)$ expression for the term in parenthesis in (3.12) is $\bar{\psi}\Gamma^{1mn}\psi\, q_n$ (our conventions for the Dirac matrices are given in the appendix), with $n = \pm, 2, \ldots 8$, which after analytic continuation (that sends also $v^i \to iv^i$) leads to

$$\mathcal{V}_3 = -2i\,v^2v_m J^{0mn}\,\partial_n G_9(\vec{b}) = 2i\,v^2 v_m(\theta\gamma^{mn}\theta)\,\partial_n G_9(\vec{b})\,, \qquad (3.13)$$

where $J^{\mu\nu\rho} \equiv \bar{\Psi}\Gamma^{\mu\nu\rho}\Psi$, Ψ is the Majorana-Weyl spinor $\Psi = \binom{\theta}{0}$, $\theta = (\eta^a, \tilde{\eta}^{\dot{a}})$ and $m, n = 1, \ldots, 9$. We have written the result also in terms of the $SO(9)$ spinor θ which is the most convenient way to connect it to matrix theory. It is clear that we can from now on restrict our attention to terms with no net power of q^+; as shown in (3.12), these terms contain enough information to reconstruct the covariant form of our amplitudes..

v^2**-term**:

$$\mathcal{V}_2 = \frac{2^2\cdot 6}{2!4!}\int \frac{d^9q}{(2\pi)^9}\frac{e^{i\vec{q}\cdot\vec{b}}}{q^2} t^{1i1jlmnp}\,v_iv_j\,\omega_{lmnp}(q,\eta)\,, \qquad (3.14)$$

where we defined

$$\omega_{i_1\dots i_{2n}}(\eta,q) \equiv \frac{1}{(2n!)}\eta_{[a_1}(\tilde{\eta}\hat{q})_{a_2}\dots\eta_{a_{2n-1}}(\tilde{\eta}\hat{q})_{a_{2n}]}\gamma^{i_1 i_2}_{a_1 a_2}\dots\gamma^{i_{2n-1}i_{2n}}_{a_{2n-1}a_{2n}} \tag{3.15}$$

and $\hat{q} = q_i\gamma^i$. It is not difficult to see that

$$t^{1i1jlmnp}v_i v_j \omega_{lmnp}(q,\eta) = \frac{2}{3}v^2(J^{1mq}J^{1n}{}_q + J^{mi\mu}J_\mu{}^{nj}\hat{v}_i\hat{v}_j)q_m\, q_n \tag{3.16}$$

with latin indices labelling the indices $i,j,\dots=\pm,2,\dots 8$, whereas greek indices run over all 10 directions. Strictly speaking the equality (3.16) holds only for $m,n=2,\dots,8$. The cases where $m,n=\pm$ are given by the terms in (3.9) with non-vanishing powers of q^+, as can be easily checked. This is the sense in which (3.16) and similar identities that will follow should be understood. We can now analytically continue the right hand side of (3.16) leading to

$$\begin{aligned}\mathcal{V}_2 &= \frac{1}{3}v^2(J^{m0q}J^n{}_{0q} + J^{m\mu}{}_i J^n{}_{\mu j}\hat{v}^i\hat{v}^j)\partial_m\partial_n G_9(\vec{b}) \\ &= -2(\theta\gamma^{pm}\theta)(\theta\gamma^{qn}\theta)v_p\, v_q\, \partial_m\partial_n G_9(\vec{b})\,,\end{aligned} \tag{3.17}$$

where the $SO(9)$ expression follows after a Fierz identity.

v^1**-term**:

$$\mathcal{V}_1 = \frac{2\cdot 20}{6!}\int\frac{d^9q}{(2\pi)^9}\frac{e^{i\vec{q}\cdot\vec{b}}}{q^2}t^{1ij_1\dots j_6}\, v_i\, \omega_{j_1\dots j_6}(q,\eta) \tag{3.18}$$

This case, as well as the next one, is a little more involved, since we have a new contribution

$$t^{1ij_1\dots j_6}v_i\omega_{j_1\dots j_6}(q,\eta) = 12\,\omega_{1ijkkj}(q,\eta) + 24\,\omega_{1jjkki}(q,\eta) - 1/2\,\epsilon^{1ij_1\dots j_6}\omega_{j_1\dots j_6}(q,\eta) \tag{3.19}$$

The first term on the right hand side of (3.19) is vanishing due to the Fierz identity (A.5). The second term is

$$\omega_{1jjkki}(q,\eta) = \frac{1}{20}J^{1nl}J_{l\mu}{}^m J^{\mu\, i\, p}q_n q_m q_p\,. \tag{3.20}$$

Using the relations (A.8) reported in the appendix, it is possible to verify that the $SO(1,9)$ expression for the ϵ-term is, up to a numerical factor

$$\epsilon_{1\mu_1\dots\mu_9}\, q^{\mu_9}\, v^{\mu_8}\, q_\alpha\, q_\beta\, J^{\mu_1\mu_2\alpha}J^{\mu_3\mu_4\beta}J^{\mu_5\mu_6\mu_7}. \tag{3.21}$$

The expression (3.21) can be brought into the same form as the right hand side of (3.20) using the identity (A.6). In order to fix the relative coefficient between the two non-vanishing contributions coming from (3.19), it is much simpler to consider the term proportional to $(q^+)^3$, in which case the spinor algebra simplifies considerably. This term is proportional to

$$-1/2\,\epsilon^{1ij_1\dots j_6}(\eta\gamma^{j_1j_2}\eta)(\eta\gamma^{j_3j_4}\eta)(\eta\gamma^{j_5j_6}\eta) + 24(\eta\gamma^{1j}\eta)(\eta\gamma^{jk}\eta)(\eta\gamma^{ki}\eta)\,. \tag{3.22}$$

By using the identity (A.7), the ϵ-term in (3.22) becomes $8(\eta\gamma^{1j}\eta)(\eta\gamma^{jk}\eta)(\eta\gamma^{ki}\eta)$. Putting all the results together we find

$$
\begin{aligned}
\mathcal{V}_1 &= \frac{4i}{45} v_i\, J^{0nl} J_{l\mu}{}^{m} J^{\mu\, i\, p}\, \partial_m \partial_n \partial_p G_9(\vec{b}) = \\
&= -\frac{4i}{9} v_i (\theta\gamma^{im}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pl}\theta)\, \partial_m \partial_n \partial_p G_9(\vec{b})\,,
\end{aligned} \tag{3.23}
$$

where the second identity in (3.23) follows from the first one by $SO(9)$ Fierz identities.

v^0**-term**:

$$
\mathcal{V}_0 = \frac{70}{8!} \int \frac{d^9 q}{(2\pi)^9} \frac{e^{i\vec{q}\cdot\vec{b}}}{q^2} t^{i_1 \ldots i_8} \omega_{i_1 \ldots j_8}(q,\eta)\,, \tag{3.24}
$$

where

$$
t^{i_1 \ldots i_8} \omega_{i_1 \ldots i_8}(q,\eta) = 24\, \omega_{ijjkklli}(q,\eta) - 1/2\, \epsilon^{i_1 \ldots i_8} \omega_{i_1 \ldots i_8}(q,\eta)\,. \tag{3.25}
$$

The $SO(1,9)$ expression for the ϵ-term is

$$
\epsilon_{\mu_1 \ldots \mu_{10}} q^{\mu_{10}} q_\alpha q_\beta q_\gamma J^{\mu_1 \mu_2 \alpha} J^{\mu_3 \mu_4 \beta} J^{\mu_5 \mu_6 \gamma} J^{\mu_7 \mu_8 \mu_9}, \tag{3.26}
$$

whereas

$$
\omega_{ijjkklli}(q,\eta) = \frac{1}{70} J^{\mu\nu m} J_{\mu}{}^{\rho n} J_{\nu}{}^{\sigma p} J_{\rho\sigma}{}^{q}\, q_m\, q_n\, q_p\, q_q\,. \tag{3.27}
$$

Again, the expression (3.26) can be cast in the form appearing on the right hand side of (3.27). By looking at the $(q^+)^4$-term, similarly to the previous case, it turns out that the ϵ-term in (3.25) gives a contribution equal to $8\,\omega_{ijjkklli}(q,\eta)$. We then obtain

$$
\begin{aligned}
\mathcal{V}_0 &= \frac{32}{8!} J^{\mu\nu m} J_{\mu}{}^{\rho n} J_{\nu}{}^{\sigma p} J_{\rho\sigma}{}^{q}\, \partial_m \partial_n \partial_p \partial_q G_9(\vec{b}) \\
&= \frac{2}{63} (\theta\gamma^{ml}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pk}\theta)(\theta\gamma^{qk}\theta)\, \partial_m \partial_n \partial_p \partial_q G_9(\vec{b})\,,
\end{aligned} \tag{3.28}
$$

where once again $SO(9)$ Fierz identities have been used to write the second identity in (3.28).

Collecting all terms we obtain the final result for the leading one-loop potential of two D0-branes [6]

$$
\begin{aligned}
\mathcal{V}^{(1)} = \Big[& v^4 + 2i\, v^2 v_m (\theta\gamma^{mn}\theta)\, \partial_n - 2 v_p\, v_q (\theta\gamma^{pm}\theta)(\theta\gamma^{qn}\theta)\, \partial_m \partial_n - \\
& -\frac{4i}{9} v_i (\theta\gamma^{im}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pl}\theta)\, \partial_m \partial_n \partial_p + \\
& + \frac{2}{63} (\theta\gamma^{ml}\theta)(\theta\gamma^{nl}\theta)(\theta\gamma^{pk}\theta)(\theta\gamma^{qk}\theta)\, \partial_m \partial_n \partial_p \partial_q \Big] G_9(\vec{b})\,.
\end{aligned} \tag{3.29}
$$

The first, second, third and last terms of (3.29) were calculated in a super Yang-Mills context in [12, 24, 25] and [26], respectively. The supersymmetry parameter θ should be identified with the spinor $\theta^3/2$ introduced in the previous section and represents the fermionic background in matrix theory.

Before concluding the present section we want to make some comments about the origin of the ϵ-terms in (3.19), (3.25). By performing an analysis of 1-point functions of massless closed string states on a disc with supercharges inserted on its boundary, it is straightforward to derive which fields exchanged between the branes are responsible for the interactions described above [27, 7, 8]. In this way, as one might expect, all the interactions, except those coming from the ϵ-terms, are due to exchange of dilatons, gravitons and Ramond-Ramond (RR) vector gauge fields. On the other hand, the ϵ-terms arise from an interesting coupling between dual RR gauge potentials, very similar to that analysed in [28, 29, 30]. In particular the ϵ-term coming from the part of the potential linear in the velocity is due to exchange of a RR one form $A_{(1)}$ and its dual seven-form $A_{(7)}$. The insertion of six supercharges corresponding to D-particles on the boundary of the disc indeed induces a non-minimal coupling to the seven-form. Schematically, in light-cone gauge it reads

$$\langle B|Q^6|A_{(7)}\rangle \sim \omega_{j_1\ldots j_6}(q)A_{[1j_1\ldots j_6]}\,, \tag{3.30}$$

where the direction 1 satisfies the Neumann boundary condition on the disc and will be identified with the time direction. The coupling (3.30) produces then an interaction

$$\langle B|Q^6|A_{(7)}\rangle\langle A_{(7)}|A_{(1)}\rangle\langle A_{(1)}|V_B|B\rangle \sim \frac{1}{q^2}v_i\,\omega_{j_1\ldots j_6}(q)\,\epsilon^{1ij_1\ldots j_6}, \tag{3.31}$$

where we used the relation between dual transverse gauge potentials $A_{(1)} = {}^*\!A_{(7)}$. Notice the relationship of this interaction to the corresponding one analysed in [30] for the D0-D6 brane system.

The ϵ-term associated to the static potential can be treated similarly. Again, the RR seven-form $A_{(7)}$ has a non-vanishing 1-point function when eight supercharges are inserted, that includes a term

$$\langle B|Q^8|A_{(7)}\rangle \sim \omega_{1j_1\ldots j_7}(q)\,A_{[j_1\ldots j_7]} \tag{3.32}$$

leading to an interaction

$$\langle B|Q^8|A_{(7)}\rangle\langle A_{(7)}|A_{(1)}\rangle\langle A_{(1)}|B\rangle \sim \frac{1}{q^2}\,\omega_{j_1\ldots j_8}(q)\,\epsilon^{j_1\ldots j_8} \tag{3.33}$$

equal to the ϵ-term appearing in the static potential. In this case the correspondence with the analogous static RR potential for the D0-D8 system, found in [29], is more subtle. Indeed, it has been shown in [29] that non-physical polarisations of the RR nine-form, identified with some RR one-form polarisations, are responsible for the RR attraction between a D0 and a D8 brane. Although these effects are clearly not visible in a transverse physical gauge, it can be seen in a covariant formalism that the RR nine-form has a non-vanishing 1-point function with D0-branes, when eight supercharges are inserted. The considerations above are of course only schematic but the key point is to highlight the presence of such interesting interactions which can be analysed in more detail along the lines of [28, 29, 30].

4. Three form scattering in matrix theory

In this section we present the results for three form scattering in matrix theory. The section will be divided into three parts. In the first, we spell out carefully the kinematics of the scattering amplitude under consideration. In the second we develop the algebra of bilinears built from the $SO(9)$ fermionic operators θ^3 acting on polarisation states. This latter development allows one to perform the matrix theory computation of S-matrix elements, once one is given the potential as in (3.29), with comparable (or even improved) efficiency to that when employing tree level Feynman diagrams in $D = 11$ supergravity. In section 3 we state our result.

4.1. Kinematics

The starting point is our matrix theory LSZ formula (2.13) which yields the S-matrix for the $1{+}2 \longrightarrow 4{+}3$ scattering of particles with momenta p^1_m, p^2_m, p^4_m, and p^3_m, respectively. To begin with, the free $U(1)$ center of mass sector of the theory decouples and yields an overall factor, $(2\pi)^9\delta^9(P'_m - P_m)\exp(-iP_mP_mT/2)$, expressing conservation of total momentum. Here $P_m = p^1_m + p^2_m$ and $P'_m = p^4_m + p^3_m$ are the total in and outgoing momenta, respectively, and the exponential is the standard factor obtained in time independent perturbation theory.[7]

A loopwise expansion of the matrix theory effective action yields

$$\Gamma(x'_m, x_m, \theta^3) = \Gamma(b_m, v_m, \theta^3) = v_m v_m T/2 + \Gamma^{(\mathrm{loops})} \tag{4.1}$$

with $v_m = (x'_m - x_m)/T$ and $b_m = (x'_m + x_m)/2$, so that the S matrix now reads (up to an overall normalisation)

$$\begin{aligned} S_{fi} = &\int d^9x'd^9x \exp(-iw_m x'_m + iu_m x_m + iv_m v_m T/2) \times \\ &\times \langle\mathcal{H}^3|\langle\mathcal{H}^4|\, e^{i\Gamma^{(\mathrm{loops})}(v_m, b_m, \theta^3)}\, |\mathcal{H}^1\rangle\, |\mathcal{H}^2\rangle\,, \end{aligned} \tag{4.2}$$

where we denoted the in and outgoing relative momenta $u_m = (p^1_m - p^2_m)/2$ and $w_m = (p^4_m - p^3_m)/2$, respectively. However, changing variables $d^9x'd^9x \to d^9(Tv)d^9b$, the integral over Tv_m may be performed, for large T, by stationary phase which yields

$$S_{fi} = e^{-i[\frac{u+w}{2}]^2\frac{T}{2}} \int d^9b\, e^{-iq_m b_m} \langle\mathcal{H}^3|\, \langle\mathcal{H}^4|\, e^{i\Gamma^{(\mathrm{loops})}(v_m=(u+w)_m/2, b_m, \theta^3)}\, |\mathcal{H}^1\rangle\, |\mathcal{H}^2\rangle\,, \tag{4.3}$$

where $\Gamma^{(\mathrm{loops})}$ is to be evaluated at $v_m = (u + w)_m/2$, which we take, henceforth, as the definition of v_m. Moreover q_m denotes the momentum transfer $q_m = w_m - u_m$. For

[7] In what follows, we shall disregard these kinematical prefactors in comparing to the SUGRA Feynman graph result since they only express the usual relation between time independent and time dependent perturbation theory.

clarity, let us give the relation between the various momenta introduced above

$$\begin{aligned} p^1_m &= P_m + u_m = P_m + v_m - q_m/2 \qquad & p^2_m &= P_m - u_m = P_m - v_m + q_m/2\,, \\ p^4_m &= P'_m + w_m = P_m + v_m + q_m/2 \qquad & p^3_m &= P'_m - w_m = P_m - v_m - q_m/2\,. \end{aligned} \tag{4.4}$$

Energy-momentum conservation fixes, in particular, the important relation $q_m v_m = 0$.

Finally, we make one last manipulation. The matrix theory effective action is an expression of the form $\Gamma^{(\text{loops})} = \int_{-T/2}^{T/2} dt\, \mathcal{V}^{(\text{loops})}[x^{\text{cl}}_m = (b_m + v_m t), \dot{x}^{\text{cl}}_m = v_m, \theta^3]$ for some potential $\mathcal{V}^{(\text{loops})}$. However, we would really like to consider a time independent potential depending on b_m and v_m and θ^3 acting on states. This is achieved by expanding the exponential inside the polarisation expectation value in (4.3) and then interchanging the dt and d^9b integrations. Then shifting $b_m \to b_m + v_m t$ and using $q_m v_m = 0$, the dt integral yields only an overall factor T (the same factor T as appearing in front of equation (3.8) in our string computation). Finally, we have the desired expression for the one-loop matrix theory S-matrix

$$S_{fi} = iT \exp\Big(-iv^2 \frac{T}{2}\Big) \int d^9 b\, e^{-iq_m b_m} \langle \mathcal{H}^3 |\, \langle \mathcal{H}^4 |\, \mathcal{V}^{(1)}(b_m, v_m, \theta^3) | \mathcal{H}^1 \rangle\, | \mathcal{H}^2 \rangle\,. \tag{4.5}$$

The loopwise expansion of the effective action is valid for large impact parameters b_m and hence small momentum transfer q_m. In section 5 we will show that this is precisely the limit dominated by t-channel tree level physics in the SUGRA computation.

4.2. Algebra of bilinears

All that remains to complete our matrix theory computation is to perform a Fourier transform with respect to b_m of (4.5) and insert the one-loop matrix theory potential (3.29) into the polarisation inner products. The result is an amplitude which may be directly compared with the result of the SUGRA tree level Feynman diagram calculation of section 5.

In our previous work, when computing the graviton-graviton scattering amplitude we used the explicit representation for the polarisation states in terms of $SO(7) \otimes U(1)$ covariant complex spinors (see [10]). Then rewriting the matrix theory potential also in terms of complex spinors, we were able to compute its polarisation expectation value. Unfortunately, if one considers now the three form polarisation states, this naive method becomes rather quickly unwieldy. Instead, we now present an $SO(9)$ covariant algebra of spinor bilinears acting on polarisation states with which amplitudes of complicated Matrix Theory potentials can be efficiently computed.

The matrix theory potential (3.29) is written solely in terms of $SO(9)$ rotation generators $\frac{1}{8}\theta^3\gamma^{mn}\theta^3$ where the operators θ^3_α satisfy the anticommutation relations $\{\theta^3_\alpha, \theta^3_\beta\} = \delta_{\alpha\beta}$. However, the states

$$|\mathcal{H}^{1,4}\rangle = \mathcal{H}^{1,4}_{\mathcal{M}}\, |-\rangle^{\mathcal{M}}_{\theta^0+\theta^3}\,, \qquad |\mathcal{H}^{2,3}\rangle = \mathcal{H}^{2,3}_{\mathcal{M}}\, |-\rangle^{\mathcal{M}}_{\theta^0-\theta^3} \tag{4.6}$$

(where the generalised index $\mathcal{M}$ denotes $\mathcal{M} \equiv \{mn\,; mnp\,; m\alpha\}$, corresponding to graviton, three-form and gravitino polarisations, respectively) are built from states depending either on the sum or difference of the centre of mass ($\{\theta^0_\alpha, \theta^0_\beta\} = \delta_{\alpha\beta}$) and Cartan fermionic coordinates which we denote as $\theta^1 = (\theta^0 + \theta^3)/\sqrt{2}$ and $\theta^2 = (\theta^0 - \theta^3)/\sqrt{2}$. Of course, the operators $\frac{1}{8}\theta^1\gamma^{mn}\theta^1$ and $\frac{1}{8}\theta^2\gamma^{mn}\theta^2$ act simply as $SO(9)$ rotation generators on the states $|-\rangle^{\mathcal{M}}_{\theta^1}$ and $|-\rangle^{\mathcal{M}}_{\theta^2}$, respectively, under which they transform in the usual fashion. Therefore, if we could write the potential only in terms of these operators the computation would be completely trivial. Yet there are cross terms since

$$\frac{1}{8}\theta^3\gamma^{mn}\theta^3 = \frac{1}{4}\theta^1\gamma^{mn}\theta^1 + \frac{1}{4}\theta^2\gamma^{mn}\theta^2 - \frac{1}{2}\theta^1\gamma^{mn}\theta^2\,. \tag{4.7}$$

Let us now concentrate on amplitudes involving bose particles only (the generalisation to the fermi case is simple but not needed here). Clearly, the difficult terms are those involving the bilinear $\theta^1\gamma^{mn}\theta^2$. However, since between bose states the expectation of an odd number of θ operators vanishes (and the 1 and 2 sectors are independent), only an even number of such operators can occur. But products $\theta^1\gamma^{mn}\theta^2\ \theta^1\gamma^{rs}\theta^2$ may always be rewritten in terms of the $SO(9)$ generators $\theta^{1,2}\gamma^{mn}\theta^{1,2}$ or three index bilinears $\theta^{1,2}\gamma^{mnp}\theta^{1,2}$ via a Fierz rearrangement, of which only the latter cause any difficulty.

Clearly, what is needed then is the action of three index operators $\theta\gamma^{mnp}\theta$ (dropping the labels 1,2) on states $|-\rangle^{\mathcal{M}}$. The result is easily obtained by making the most general ansatz and fixing the coefficients via the explicit representation given in [10], we find

$$\theta\gamma^{mnp}\theta|-\rangle^{tu} = -i24\sqrt{3}(\delta^{mt}\,|-\rangle^{npu} - \tfrac{1}{9}\,\delta^{tu}\,|-\rangle^{mnp})\,, \tag{4.8}$$

$$\theta\gamma^{mnp}\theta|-\rangle^{tuv} = i24\sqrt{3}\,\delta^{mt}\delta^{nu}\,|-\rangle^{pv} + \tfrac{2i}{3}\,\epsilon^{mnptuvwyz}|-\rangle^{wyz}. \tag{4.9}$$

On the right hand side of (4.8) and (4.9) one must (anti)symmetrize (with unit weight) over all indices according to the symmetry properties of the left hand sides of these equations.

4.3. Results

Given the algebra (4.8) and (4.9) along with the potential (3.29), only a moderate amount of computer algebra [31] is now required to obtain the Matrix Theory one-loop three form–three form eikonal scattering amplitude. Our result consists of 103 terms and is given by (normalising the v^4 term to unity)

$$\begin{aligned}\mathcal{A} = \frac{1}{q^2}\Big\{&\tfrac{1}{2}\,v^4\,C_{14}C_{23} + 3v^2\Big(C_{14}(v,q)C_{23} - C_{14}(q,v)C_{23}\Big)\\
&+v^2\Big(\tfrac{1}{2}\,\tfrac{3}{4}\,C_{12}(q,q)C_{34} + \tfrac{1}{2}\,\tfrac{3}{4}\,C_{12}C_{34}(q,q) - \tfrac{3}{2}\,C_{14}(q,q)C_{23} - \tfrac{3}{4}\,C_{13}(q,q)C_{24}\\
&+\tfrac{1}{2}\,\tfrac{9}{4}\,C_{14}(q,m_1)C_{23}(q,m_1) + \tfrac{1}{2}\,\tfrac{9}{4}\,C_{14}(m_1,q)C_{23}(m_1,q) - \tfrac{9}{4}\,C_{14}(q,m_1)C_{23}(m_1,q)\\
&+\tfrac{1}{2}\,\tfrac{9}{4}\,C_{13}(q,m_1)C_{24}(q,m_1) + \tfrac{1}{2}\,\tfrac{9}{4}\,C_{13}(m_1,q)C_{24}(m_1,q) - \tfrac{9}{4}\,C_{12}(q,m_1)C_{34}(q,m_1)\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}\tfrac{9}{2}\,C_{12}(q,m_1,q,m_2)C_{34}(m_1,m_2)+\tfrac{1}{2}\tfrac{9}{2}\,C_{12}(m_1,m_2)C_{34}(q,m_1,q,m_2)\\
&-\tfrac{9}{2}\,C_{13}(q,m_1,q,m_2)C_{24}(m_1,m_2)\Big)\\
&-\tfrac{1}{2}\tfrac{9}{2}\,C_{12}(v,v)C_{34}(q,q)+\tfrac{9}{2}\,C_{12}(v,q)C_{34}(v,q)+\tfrac{1}{2}\tfrac{9}{2}\,C_{14}(v,q)C_{23}(v,q)\\
&-\tfrac{9}{2}\,C_{14}(v,q)C_{23}(q,v)+\tfrac{1}{2}\tfrac{9}{2}\,C_{14}(q,v)C_{23}(q,v)+\tfrac{9}{2}\,C_{13}(v,v)C_{24}(q,q)\\
&-\tfrac{1}{2}\tfrac{9}{2}\,C_{13}(v,q)C_{24}(v,q)-\tfrac{1}{2}\tfrac{9}{2}\,C_{13}(q,v)C_{24}(q,v)-\tfrac{1}{2}\tfrac{9}{2}\,C_{34}(v,v)C_{12}(q,q)\\
&+\tfrac{1}{2}\,3C_{12}(v,q,v,q)C_{34}+\tfrac{1}{2}\,3C_{34}(v,q,v,q)C_{12}\\
&-6C_{14}(v,q,v,q)C_{23}-3C_{13}(v,q,v,q)C_{24}\\
&-\tfrac{1}{2}\,9C_{12}(m_1,m_2)C_3(v,q,m_1)C_4(v,q,m_2)-\tfrac{1}{2}\,9C_{34}(m_1,m_2)C_1(v,q,m_1)C_2(v,q,m_2)\\
&+9C_{24}(m_1,m_2)C_1(v,q,m_1)C_3(v,q,m_2)\\
&-9C_{12}(v,q,v,m_1)C_{34}(q,m_1)+9C_{12}(v,q,q,m_1)C_{34}(v,m_1)\\
&+9C_{13}(v,q,v,m_1)C_{24}(q,m_1)-9C_{13}(v,q,q,m_1)C_{24}(v,m_1)\\
&-9C_{13}(m_1,v)C_{24}(q,m_1,v,q)+9C_{13}(m_1,q)C_{24}(v,m_1,v,q)\\
&+9C_{12}(m_1,v)C_{34}(q,m_1,v,q)-9C_{12}(m_1,q)C_{34}(v,m_1,v,q)\\
&+\tfrac{1}{2}\,18C_{12}(v,m_1,v,m_2)C_{34}(q,m_1,q,m_2)+\tfrac{1}{2}\,18C_{34}(v,m_1,v,m_2)C_{12}(q,m_1,q,m_2)\\
&-18C_{12}(v,m_1,q,m_2)C_{34}(v,m_1,q,m_2)\\
&+\tfrac{9}{4}\,C_{12}(v,q)C_{34}(q,q)-\tfrac{9}{4}\,C_{14}(v,q)C_{23}(q,q)+\tfrac{9}{4}\,C_{14}(q,v)C_{23}(q,q)\\
&+\tfrac{9}{4}\,C_{13}(v,q)C_{24}(q,q)-\tfrac{9}{4}\,C_{13}(q,v)C_{24}(q,q)-\tfrac{9}{4}\,C_{34}(v,q)C_{12}(q,q)\\
&+\tfrac{9}{2}\,C_{12}(v,q,q,m_1)C_{34}(q,m_1)-\tfrac{9}{2}\,C_{14}(v,q,q,m_1)C_{23}(q,m_1)\\
&+\tfrac{9}{2}\,C_{14}(v,q,q,m_1)C_{23}(m_1,q)+\tfrac{9}{2}\,C_{13}(v,q,q,m_1)C_{24}(q,m_1)\\
&-\tfrac{9}{2}\,C_{24}(q,m_1,v,q)C_{13}(m_1,q)-\tfrac{9}{2}\,C_{14}(q,m_1,v,q)C_{23}(q,m_1)\\
&+\tfrac{9}{2}\,C_{14}(q,m_1,v,q)C_{23}(m_1,q)-\tfrac{9}{2}\,C_{34}(q,m_1,v,q)C_{12}(m_1,q)\\
&-9C_{12}(v,m_1,q,m_2)C_{34}(q,m_1,q,m_2)+9C_{34}(v,m_1,q,m_2)C_{12}(q,m_1,q,m_2)\\
&-\tfrac{1}{2}\tfrac{9}{8}\,C_{12}(q,q)C_{34}(q,q)-\tfrac{1}{2}\tfrac{9}{8}\,C_{14}(q,q)C_{23}(q,q)-\tfrac{1}{2}\tfrac{9}{8}\,C_{13}(q,q)C_{24}(q,q)\\
&+\tfrac{1}{2}\tfrac{9}{2}\,C_{12}(q,m_1,q,m_2)C_{34}(q,m_1,q,m_2)+\tfrac{1}{2}\tfrac{9}{2}\,C_{12}(q,m_1,q,m_2)C_{34}(q,m_2,q,m_1)\\
&+\tfrac{1}{2}\tfrac{9}{2}\,C_{13}(q,m_1,q,m_2)C_{24}(q,m_2,q,m_1)\\
&+\Big[C_1\longleftrightarrow C_2\,,\,C_3\longleftrightarrow C_4\Big]\Big\}\,, \qquad (4.10)
\end{aligned}
$$

where we have introduced the notation in which an n-index tensor $T_{m_1\ldots m_n}$ written as a function of a vector q_m denotes $T(q,m_2,\ldots,m_n)\equiv T_{m_1m_2\ldots m_n}q_{m_1}$. Moreover the tensors C_{ij} $(i,j=1,\ldots,4)$ denote the contraction of polarisation tensors, for example $C_{14}(q,v)=C_1^{m_1m_2m_3}C_4^{m_3m_2n_1}q^{m_1}v^{\nu_1}$ (the indices are contracted between the two tensors in the order indicated, another example is $C_{12}=C_1^{m_1m_2m_3}C_2^{m_3m_2m_1}$). In the next section we shall see that this result[8] yields perfect agreement with SUGRA Feynman graphs.

[8] We note, in passing, that in obtaining (4.10), products of as many as four nine dimensional Levi

5. The supergravity computation

The bosonic sector of 11-dimensional supergravity [1] is given by

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{2\kappa^2}\sqrt{-g}\,R - \tfrac{1}{8}\sqrt{-g}\,(F_{MNPQ})^2 \\ &-\tfrac{\sqrt{3}}{12^3\kappa}\,\varepsilon^{M_1\ldots M_{11}}F_{M_1M_2M_3M_4}\,F_{M_5M_6M_7M_8}\,C_{M_9M_{10}M_{11}}\,,\end{aligned} \tag{5.1}$$

where $F_{MNPQ} = 4\partial_{[M}C_{NPQ]}$ and $g = \det g_{MN}$. Perturbative quantum gravity may be studied by considering small fluctuations h_{MN} from the flat metric η_{MN}

$$g_{MN} = \eta_{MN} + \kappa\, h_{MN}\,, \tag{5.2}$$

where κ is the 11-dimensional gravitational coupling constant. From now on we raise and lower indices with the flat metric η_{MN}. Propagators are obtained in the usual fashion. For the graviton we employ the harmonic (de Donder) gauge $\partial_N h^N{}_M - (1/2)\partial_M h^N{}_N = 0$ with propagator (in d dimensions)

$$\langle h_{MN}(k)h_{PQ}(-k)\rangle = \frac{-4i}{k^2}\Big(\eta_{(M|P|}\eta_{N)Q} - \frac{1}{d-2}\eta_{MN}\eta_{PQ}\Big)\,. \tag{5.3}$$

For the antisymmetric tensor, the gauge fixing function $\partial_M C^M{}_{NP}$ in the weighted gauge $\mathcal{L}_{\rm fix} = -\frac{3}{2}(\partial^M C_{MNP})^2$ yields the Feynman propagator

$$\langle C_{M_1M_2M_3}(k)\,C^{N_1N_2N_3}(-k)\rangle = \frac{-i}{k^2}\,\delta^{N_1}_{[M_1}\,\delta^{N_2}_{M_2}\,\delta^{N_3}_{M_3]}\,. \tag{5.4}$$

The relevant vertices are easily read off from (5.1). In particular, note that h_{MN} couples in the usual way to the three form stress-energy tensor.

At tree level, the only graphs contributing to four point, three form scattering are the single graviton and three form exchange diagrams. These are easily computed and in the $t = -2p^1_M p^M_4$ channel, which, as we shall see dominates eikonal physics (the s and u channels follow anyway by Bose symmetry) one finds

$$\begin{aligned}\mathcal{A}_{\mathbf{84}} = \frac{-i\kappa^2}{R}\frac{1}{t}\Big\{&\frac{3}{18^2}(\epsilon F^1F^4)^{MNP}(\epsilon F^2F^4)_{MNP} - \\ &-32\Big[\Big(\frac{d+16}{d-2}\Big)(F^2\cdot F^3)(F^1\cdot F^4) - 32(F^2\cdot F^3)_{(AB)}(F^1\cdot F^4)^{AB}\Big]\Big\}\,.\end{aligned} \tag{5.5}$$

The factor R in (5.5) represents the radius of the compactified light-like circle, $F^i_{MNPQ} = p^i_{[M}C^i_{NPQ]}$ $(i = 1,\ldots,4)$ is the curl of the eleven dimensional polarisation tensor C^i_{MNP} and we denote $F^i\cdot F^j = F^i_{MNOP}F_j^{MNOP}$, $(F^i\cdot F^j)_{AB} = F^i_{AMNP}F^{j\,MNP}_B$ and $(\epsilon F^iF^j)^{MNP} = \epsilon^{MNPM_1\ldots M_4N_1\ldots N_4}F^i_{M_1\ldots M_4}F^j_{N_1\cdots N_4}$. Of course, one must put $D = 11$ in

---

Civita symbols were encountered and expanded in Kronecker deltas. For convenience, we have also put an explicit $\frac{1}{2}$ in front of terms mapped to themselves under the replacement $[1 \longleftrightarrow 2\,, 3 \longleftrightarrow 4]$.

this formula. The momenta and polarisations satisfy the mass shell and gauge conditions $p^i_M p^M_i = 0$ and $p^M_i C^i_{MNP} = 0$, respectively.

Since we are not aware of any other occurrences of the result (5.5) in the literature, for added certainty we considered the dimensional reduction of the amplitude (5.5) to ten dimensions setting $p^i_{11} = 0$. In this case, the three form gauge field, that transforms as an **84** with respect to the little group $SO(9)$, splits into **56** and **28** representations of $SO(8)$, that is the little group in ten dimensions. Then, we checked the validity of (5.5) by computing the four-point function of the Ramond-Ramond three form (**56**) and of the antisymmetric tensor field (**28**) in type IIA string theory. By taking then the low-energy $\alpha' \to 0$ limit of the IIA amplitudes we obtained precisely the dimensional reduction of (5.5) to ten dimensions.

5.1. Light cone supergravity amplitudes

In order to make a comparison with the matrix theory results we need to rewrite our supergravity t channel amplitude in terms of physical transverse nine dimensional degrees of freedom. Since we are considering the $N = 2$ Discrete Light Cone Quantisation (DLCQ) formulation [9] of the theory, we work in light cone coordinates and specialise to the case of vanishing p^- momentum exchange. Define, therefore,[9]

$$p^\pm = p_\mp = \frac{p^{10} \pm p^0}{\sqrt{2}}\,, \tag{5.6}$$

so that on-shell momenta satisfy $0 = p^M p_M = 2p^+p^- + p_m p_m$ where nine dimensional indices $m, n, \ldots$ are contracted with a Kronecker delta. The gauge condition $p^M C_{MNP} = 0$ may then be solved in terms of 84 physical polarisations C_{mnp} via

$$C_{+-m} = 0 = C_{+mn}\,,\; C_{-mn} = -\frac{1}{p^-} p_r C_{rmn}\,, \tag{5.7}$$

where we have used the residual gauge freedom to set $C_{+mn} = 0$. From now on, we measure momenta in units of the compactified radius R so that $p^- = 1$ and $p^+ = -\frac{1}{2}p^m p^m$.

The $N = 2$ DLCQ matrix theory describes $1 + 2 \longrightarrow 4 + 3$ scattering in the case of vanishing p^- momentum exchange so we may therefore write the incoming and outgoing momenta as

$$\begin{aligned}
p^1_M &= (-\tfrac{1}{2}\,(v_m - q_m/2)^2, 1, v_m - q_m/2)\,,\\
p^2_M &= (-\tfrac{1}{2}\,(v_m - q_m/2)^2, 1, -v_m + q_m/2)\,,\\
p^4_M &= (-\tfrac{1}{2}\,(v_m + q_m/2)^2, 1, v_m + q_m/2)\,,\\
p^3_M &= (-\tfrac{1}{2}\,(v_m + q_m/2)^2, 1, -v_m - q_m/2)\,,
\end{aligned} \tag{5.8}$$

[9] Notice the conventions used here are slightly different from those used in section 3.

where we have set the nine dimensional centre of mass momentum to zero by transverse Galilean invariance. Importantly, note that conservation of p^+ momentum implies $v_m q_m = 0$. The Mandelstam variables in this parametrisation read

$$t = q_m^2 = -2p_M^1 p_4^M\,, \quad s = 4v_m^2 + q_m^2 = -2p_M^1 p_2^M \quad u = 4v_m^2 = -2p_M^1 p_2^M = s - t\,. \quad (5.9)$$

Eikonal scattering, for which the scattering angle $\mathrm{Cos}^{-1}([v^2 - q^2/4]/[v^2 + q^2/4]) \sim \sqrt{q^2/v^2}$ is small takes place for small $q^2 = t$. Therefore we must study the amplitude (5.5) in the small t limit which is dominated by the t-pole. In fact, we argued above (see section 3) that our D0-brane computation was not reliable for contact terms proportional to q^2, and therefore in this work we only study those terms which do not cancel the t-pole (one may therefore disregard all terms with an s or u pole). We will further discuss this restriction in the conclusion.

It only remains now to state our main result, namely that substituting equations (5.7), (5.8) and (5.9) into the amplitude (5.5) and neglecting terms cancelling the t-pole, one reproduces precisely the 103 terms of the matrix theory amplitude (4.10).

6. Conclusions

In this paper we have computed a four-point scattering amplitude and shown, analogously to the graviton case analyzed in [6], that matrix theory also reproduces correctly the tensorial structures of the $D = 11$ SUGRA three form couplings. Both the present and our previous matrix computation have been performed by considering only the leading one-loop D0-brane effective potential that is protected by supersymmetry from quantum corrections [32]. Therefore the agreement found might have been expected as is the case for most of the one-loop phase shift calculations appearing in the literature. Furthermore, in principle, it should be possible to fix the structure and coefficients of equation (3.29) by supersymmetry alone [32]. However, without a formalism in which amplitudes can be derived from the matrix potential (3.29), it would have remained unclear how such a potential could be compared with the tensorial structure of the SUGRA amplitude. Clearly our work resolves this issue and renders the precise relationship between supersymmetry in each of these models more transparent. We also remark that our formalism might provide a route to establishing eleven dimensional Lorentz invariance of matrix theory.

Another issue deserving comment is the choice of background about which one perturbatively expands $D = 11$ SUGRA in order to compare with matrix theory. For the case of finite (and actually small) N, it is clearly natural to expand about a flat background and ignore the geometry induced by the D-particles themselves (since in that case these states appear as fundamental Kaluza-Klein states, i.e., excitations of the flat vacuum), in contrast to the large N case when they can be represented as classical sources, modifying then the background geometry.

Throughout this paper we have considered t-channel amplitudes only. Since we consider quantum asymptotic states which can describe in and outgoing particles scattering at any angle, this restriction can in principle be relaxed, although it is not completely clear how to compute the transition element (2.9) (or equivalently the corresponding path-integral (2.12)) in this case.[10] Furthermore, our current work is also limited to the case of vanishing p^- momentum exchange; although of considerable interest, interactions involving p^- exchange are clearly not visible in our perturbative matrix theory framework.

Finally, an important line of development could be to use our technique to analyse higher order matrix theory scattering amplitudes that would correspond to, say, one-loop effects in supergravity. This kind of comparison will put matrix theory to a much more stringent test and will be crucial in trying to understand the range of validity of the theory itself.

Acknowledgments

We wish to thank C.A. Scrucca for important discussions and correspondence. In our computations we have made extensive use of the computer algebra sytem FORM [31]. This work was supported in part by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).

A. Conventions and Fierz identities

We employ the following Dirac matrix conventions, the 32×32 $SO(1,9)$ matrices are

$$\Gamma^0 = \begin{pmatrix} 0 & I_{(16)} \\ -I_{(16)} & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i_{(16)} \\ \gamma^i_{(16)} & 0 \end{pmatrix}, \quad i = 1, ..., 9\,, \tag{A.1}$$

where the $SO(9)$ Dirac matrices are chosen to be real and symmetric

$$\gamma^9_{(16)} = \begin{pmatrix} I_{(8)} & 0 \\ 0 & -I_{(8)} \end{pmatrix}, \quad \gamma^i_{(16)} = \begin{pmatrix} 0 & \gamma^i_{(8)} \\ (\gamma^i_{(8)})^{\mathrm{T}} & 0 \end{pmatrix}, \quad i = 1, ..., 8\,. \tag{A.2}$$

The $SO(8)$ Dirac matrices $\gamma^i_{(8)}$ are the same as those appearing in volume one of [23]. The charge conjugation matrix C is identified with Γ^0 so that for a Majorana spinor $\bar{\Psi} = \Psi^{\mathrm{T}}\Gamma^0$. In section 3 use has been made of the following Fierz identities

$$\bar{\Psi}\Gamma^{\mu\nu\rho}\Psi \; \bar{\Psi}\Gamma_{\mu\nu}{}^{\sigma}\Psi = 0\,, \tag{A.3}$$

[10] In the context of classical gravity source-probe approach, recoil effects have been recently taken into account in [33].

which in $SO(9)$ and $SO(8)$ language read

$$(\theta\gamma^{ijk}_{(16)}\theta)(\theta\gamma^{ijl}_{(16)}\theta) - 2(\theta\gamma^{ik}_{(16)}\theta)(\theta\gamma^{il}_{(16)}\theta) = 0\,, \qquad (A.4)$$

$$(\eta\gamma^{ijk}_{(8)}\tilde{\eta})(\eta\gamma^{ijl}_{(8)}\tilde{\eta}) - 2(\eta\gamma^{k}_{(8)}\tilde{\eta})(\eta\gamma^{l}_{(8)}\tilde{\eta}) - (\eta\gamma^{ik}_{(8)}\eta)(\tilde{\eta}\gamma^{il}_{(8)}\tilde{\eta}) - (\eta\gamma^{il}_{(8)}\eta)(\tilde{\eta}\gamma^{ik}_{(8)}\tilde{\eta}) = 0 \qquad (A.5)$$

with $\Psi = \binom{\theta}{0}$, $\theta = (\eta^a, \tilde{\eta}^{\dot{a}})$ and the common notation $\gamma^{i_1\cdots i_n} \equiv (1/n!)\,\gamma^{i_1}...\gamma^{i_n} \pm$ perm. valid for all gamma matrices.

The expressions (3.21), (3.26) reduce to the form appearing in the right hand sides of equations (3.20) and (3.27), using the relation

$$q_\alpha q_\beta \bar{\Psi}\Gamma^{\mu_1\mu_2\alpha}\Psi\ \bar{\Psi}\Gamma^{\mu_3\mu_4\beta}\Psi \sim q_\alpha q_\beta\ \bar{\Psi}\Gamma^{\alpha}{}_{\mu\nu}\Psi\ \bar{\Psi}\Gamma^{\mu_1\mu_2\mu_3\mu_4\beta\mu\nu}\Psi + \dots\,, \qquad (A.6)$$

where dots stand for terms proportional to $\delta^{\mu_i}_{\alpha}, \delta^{\mu_i}_{\beta}, \delta^{\mu_i}_{\mu_j}$ that vanish in (3.21), (3.26) and contact terms proportional to δ^{α}_{β} have been neglected. In order to write our potential in the form shown in (3.29), the $SO(9)$ Fierz identity (A.5) of [26] as well as (A.4) are needed. We used moreover the Fierz identity

$$(\eta\gamma^{[ij}\eta)(\tilde{\eta}\gamma^{kl]}\tilde{\eta}) = -\frac{1}{24}\epsilon^{ijklmnpq}(\eta\gamma^{mn}\eta)(\tilde{\eta}\gamma^{pq}\tilde{\eta}) \qquad (A.7)$$

as well as those involving the ϵ tensor

$$\begin{aligned}
(\mathbf{A})\cdot q_{i_1}(\eta\gamma^{i_2}\tilde{\eta})(\tilde{\eta}\gamma^{i_3i_4m}\eta) &\sim (\mathbf{A})\cdot q_{i_1}\left[(\eta\gamma^{i_2m}\eta)(\tilde{\eta}\gamma^{i_3i_4}\tilde{\eta}) - (\eta\gamma^{i_3i_4}\eta)(\tilde{\eta}\gamma^{i_2m}\tilde{\eta})\right],\\
(\mathbf{A})\cdot q_k(\eta\gamma^{i_1i_2k}\tilde{\eta})(\tilde{\eta}\gamma^{i_3i_4m}\eta) &\sim (\mathbf{A})\cdot q_{i_4}\left[(\eta\gamma^{i_1i_2}\eta)(\tilde{\eta}\gamma^{mi_3}\tilde{\eta}) + (\eta\gamma^{mi_3}\eta)(\tilde{\eta}\gamma^{i_1i_2}\tilde{\eta})\right],\\
(\mathbf{A})\cdot q_{i_1}(\eta\gamma^{m}\tilde{\eta})(\tilde{\eta}\gamma^{i_2i_3i_4}\eta) &\sim (\mathbf{A})\cdot q_{i_1}\left[(\eta\gamma^{i_2m}\eta)(\tilde{\eta}\gamma^{i_3i_4}\tilde{\eta}) - (\eta\gamma^{i_3i_4}\eta)(\tilde{\eta}\gamma^{i_2m}\tilde{\eta})\right],\\
(\mathbf{B})\cdot(\tilde{\eta}\gamma^{i_2i_3m}\eta)(\eta\gamma^{ni_4}\eta) &\sim (\mathbf{B})\cdot(\eta\gamma^{nl}\eta)(\eta\gamma^{i_2i_3i_4lm}\tilde{\eta})\,, \qquad (A.8)\\
\epsilon^{1ii_1\dots i_6}\,v_i\,q_{i_1}\,q_m\,q_n(\eta\gamma^{i_6n}\eta)(\eta\gamma^{i_2i_3i_4}\tilde{\eta})(\tilde{\eta}\gamma^{i_5m}\tilde{\eta}) &\sim\\
\epsilon^{1ii_1\dots i_6}\,v_i\,q_{i_1}\,q_m\,q_n&\left[(\tilde{\eta}\gamma^{lm}\tilde{\eta})(\eta\gamma^{i_2i_3i_4i_5l}\tilde{\eta}) + 2(\tilde{\eta}\gamma^{i_5l}\tilde{\eta})(\eta\gamma^{i_2i_3i_4lm}\tilde{\eta})\right],\\
\epsilon^{i_1\dots i_8}q_{i_1}q_mq_nq_k(\eta\gamma^{i_2m}\eta)(\tilde{\eta}\gamma^{i_3n}\tilde{\eta})(\eta\gamma^{i_4i_5k}\tilde{\eta})(\tilde{\eta}\gamma^{i_6i_7i_8}\eta) &\sim\\
\epsilon^{i_1\dots i_8}q_{i_1}q_mq_nq_k(\eta\gamma^{i_2m}\eta)(\tilde{\eta}\gamma^{i_3n}\tilde{\eta})&\left[(\tilde{\eta}\gamma^{kl}\tilde{\eta})(\eta\gamma^{li_4\dots i_8}\tilde{\eta}) + (\tilde{\eta}\gamma^{li_8}\tilde{\eta})(\eta\gamma^{i_4\dots i_7kl}\tilde{\eta})\right],
\end{aligned}$$

where

$$(\mathbf{A}) = \begin{cases} \epsilon^{1ii_1\dots i_6}\,v_i\,q_m\,q_n\,(\eta\gamma^{i_5i_6n}\tilde{\eta}) \\ \epsilon^{i_1\dots i_8}\,q_m\,q_n\,q_p\,(\eta\gamma^{i_5i_6n}\tilde{\eta})(\eta\gamma^{i_7i_8p}\tilde{\eta}) \end{cases}$$

$$(\mathbf{B}) = \begin{cases} \epsilon^{1ii_1\dots i_6}\,v_i\,q_{i_1}\,q_n\,q_m\,(\tilde{\eta}\gamma^{i_5i_6}\tilde{\eta}) \\ \epsilon^{i_1\dots i_8}\,q_{i_1}\,q_p\,q_n\,q_m\,(\tilde{\eta}\gamma^{i_5i_6}\tilde{\eta})(\eta\gamma^{i_7i_8p}\tilde{\eta}) \end{cases}$$

valid, respectively, for the linear in v and static term in the potential. As explained in section 3, in order to fix the relative factors between the two contributions appearing in the linear in v and static effective potential, there is no need to know the coefficients in (A.6) and (A.8), as it is by far more convenient to use only (A.7).

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